Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

SKETCH OF THE ORIGIN AND DEVELOPMENT OF GEOMETRY PRIOR TO 1850.[1]

ALL phases of culture are so interconnected that it would indeed be in vain to study any branch of history, beginning with a determinate epoch, without throwing a glance over anterior times and events.[2] If this historical maxim is difficult to confute with regard to any one of the sciences known to us, it seems endowed with irrefragable truth when applied to a discipline so conservative as mathematics, which does not destroy the labors of preceding periods to construct in their stead new edifices.[3] Consequently before entering upon the subject proper of this history, that is, before speaking of modern geometry, we will examine briefly the origins of our science and the stages of development it passed through before reaching the state beginning with which we propose to study minutely its development.

1. To determine and historically demonstrate the first origin of geometrical research is a question which we should attempt in vain to resolve,[4] because no written document enables us to be present at the awakening and first stammering of the human intelligence. The every-day experiences of any intelligent person lead in so natural a manner to the conception of the simplest geometrical forms, and to the consideration of their mutual relations,

---

[1] Translated from the Italian of Loria by Dr. George Bruce Halsted.

[2] Libri, *Histoire des sciences mathématiques en Italie*, 1 (Paris, 1838), p. 3.

[3] Hankel, *Die Entwickelung der Mathematik in den letzten Jahrhunderten* (2. Aufl., Tübingen, 1885), p. 7.

[4] This is treated from the psychological point of view in Book I. of the *Histoire des mathématiques* of Hoefer (2d ed., Paris, 1879), which we cite because curious, not to give assent to the ideas there maintained.

that we should seek in vain to assign the epoch of the birth of geometry, and still less are we able to cite the man to whom belongs the honor of having first cultivated this noble discipline. The information which the ancients have transmitted to us on this point is so uncertain, vague, and untrustworthy, that, if not complete darkness, certainly only glimmering light meets him who proposes to explain how and why geometry entered to make part of the field of investigation for thinkers; nay such light is so scarce that it only permits us to see the outlines of certain fragments of greater mass, which have withstood the injuries of time. What these are and what value they possess we will now briefly tell, occupying ourselves successively with the various peoples that in most remote antiquity reached a stage of civilisation sufficient to permit the study of pure science.

We shall not speak of the Etruscans, in the first place because the information relative to the geometrical thought of a people whose language is even now an enigma for philologists is, if not null, certainly very slight, and in the second place because it seems certain that they exercised influence only on the Romans, who were started and directed by them in geodetic research (see § 6). If we are equally silent about the Hindoos, it is because not being able to determine with certainty whence came the information about them which we now have at our disposal, we have no right to speak of a Hindoo geometry more ancient than and therefore independent of the Greek Geometry, with which shortly we shall be occupied. In the same way we shall deal with the Chinese; for who could ever be sure that the geometric knowledge (in any case very limited) possessed by them was not imported from abroad and then declared national property by a people whose vanity is such that even to-day it boasts of having reached in the most remote antiquity inventions at which other nations have with difficulty arrived in recent times?

The people which had its original home on the banks of the Tigris and Euphrates is held to be the first of all to cultivate Arithmetic and Astronomy. Often also, among the Assyrians and Babylonians, one can trace some knowledge of geometry, a science toward which they felt themselves drawn by the hope of being able to apply it to foretell the future; and it is perhaps from their having labored with *geomancy* that they were led to consider, as they did, parallels, triangles, and quadrangles, to seek for methods to practically construct right angles, to divide the circumference of a circle into 6 and 360 equal parts, to determine an approximate value for the ratio of the circumference of a circle to its diameter.

2. Of far more importance for us than the people just mentioned were the inhabitants of the banks of the Nile, for the Greeks, who are without doubt to be considered our scientific progenitors, declared themselves, and were in fact, debtors to them for the foundation on which they erected their geometric edifice. Herodotus, who travelled in Egypt about 460 B. C., asserts that geometry had its origin in that country when King Sesostris divided equally among his subjects all the arable land in his realm; adding that a very powerful impulse for occupying themselves with geometry was given the Egyptians from the necessity to restore every year the boundary lines between the various properties, which lines the Nile obliterated during its periodical inundations. It is therefore not surprising to find that also Isocrates, about 393 B. C., declares the study of geometry was recommended to the Egyptian youth. In a similar manner Plato and Aristotle, Diodorus Siculus and Heron of Alexandria, Strabo and Democritus express themselves; nay, this last author, to show his own geometric ability, finds nothing better to assert than: "In the construction of lines by means of conclusions drawn from hypotheses, no one has surpassed me, not even the so-called ἁρπεδονάπται of the Egyptians." Who these "harpedonaptai" were is not known with absolute certainty, yet the hypothesis of M. Cantor[1] seems to us quite plausible. According to him their name (rope-stretchers) was derived from its having been their principal duty to construct in practice right angles, which they did by forming into a right triangle a rope divided into three parts respectively 3, 4, and 5 units in length.

---

[1] *Vorlesungen über Geschichte der Math.* (Leipsic, 1st ed., 1880, Vol. I., p. 55, 2nd ed., 1894, p 64).

The practical origin assigned to geometry by Herodotus, the special operation assigned to the "harpedonaptai," and the existence in Egypt of huge and wonderful architectural and hydraulic works, lead us to believe that the geometry of the Egyptians had a particular tendency towards applications, and this supposition is strengthened by the examination of a very important written document which we possess, regarding the mathematics of this people. We allude to the *Manual of the Calculator* preserved for us in the now celebrated Papyrus Rhind, written by a certain Ahmes 17 or 20 centuries B. C.[1] Unfortunately this writing, while offering a large mass of matter concerning the ancient methods of calculation, gives scant information regarding the geometric knowledge possessed by the inhabitants of the land of the Pharaohs, for geometry is not treated there *ex-professo,* but certain rules, given for evaluating surfaces and volumes, are tacitly applied to illustrate arithmetical procedures. In this *Manual* are found therefore only a few traces of metric geometry, upon the value of which there is not perfect agreement among the various historians.[2] Thus it seems indisputable that the Egyptians knew how to calculate the area of a square, but it is not certain that they knew how to do the same for a rectangle, and still more doubtful if they were able to measure with exactness the area of a triangle or a trapezoid. Again they make a circle equal to a square whose side is 8/9 of the diameter, which is the same as assuming $\pi = (16/9)^2 = 3.1604$,—a fair approximation. As for the rules by which they measured the capacity of certain granaries, it is impossible to judge, the form of these not being told; however it helps decidedly to suppose that certain problems concerning the pyramids were solved by them by a special calculus[3] presumably analogous to that which to-day we use in availing ourselves of the notions of *sine* and *tangent* of an angle. It is seen therefore that the geometry of the ancient Egyptians offers several obscure points, to clear up which would require recourse to other documents, but these are either still unknown or have not yet been deciphered.[1] Meanwhile, until it has been demonstrated that the Egyptians knew how to explain the geometric rules applied by Ahmes, we feel authorised to deny to the collections of geometrical facts which they possessed the character of a true science.

---

[1] A. Eisenlohr, *Ein mathematisches Handbuch der alten Aegypter* (Leipsic, 1st ed., 1877, 2nd ed., 1891).

[2] See besides the works already cited: E. and V. Revillout in *Revue Égyptologique*, 2, 1881, pp. 304–314, and Em. Weyr, *Ueber die Geometrie der alten Aegypter* (Vienna, 1884).

[3] M. Cantor, "Ueber den sogenannten Seqt der alten Aegypter" (*Wiener Ber.*, 90, 1884).

3. Such conclusions have importance principally because (see § 1) they exempt us from treating the question of the nature and extent of the foreign influence on Greek geometry.[2] In fact, admitting that the Babylonians and Egyptians had communicated to the Greeks all their geometric knowledge, these would in consequence have received from them only a very small amount, and certainly not the stimulus to mathematical research; moreover this is confirmed by the unity of direction in the evolution of all the Greek geometry, a unity which would be with difficulty explained by one considering it as a tree transplanted from a foreign land.

The person by whose initiative in Greece the torch of science is lighted and glimmers fitfully, is Thales of Miletus.[3] To him we are debtors for the transportation into Europe of the germs of the exact sciences, and for the first attempts to cultivate them. If to him and to his followers (the members of the Ionic school) we cannot assign any important mathematical discovery, it is because the indisputable tendency toward physical research which Thales possessed became so pronounced in his disciples and successors (Anaximander and Anaximenes) that these came to disregard the investigations of pure mathematics. Thales and the Ionic school represent therefore, in our opinion, the twilight, preceding the dawn of Greek mathematics.

---

[1] Cantor, *Vorlesungen*, 1 (2nd ed.), p. 23.

[2] As is known, the analogous question for the other disciplines and especially as to philosophy is extremely controverted: see Zeller, *Die Philosophie der Griechen* (Leipsic).

[3] Further particulars concerning the geometers of whom the present paragraph treats (specially as to the relative chronological determinations) will be found in the monograph by the author: *Le scienze esatte nell' antica Grecia*, Libro I, *I geometri greci precursori di Euclide* (*Mem. della R. Accademia di Modena*, II., 10, 1893).

But the sect of Ionic physicists having disappeared, another man arises and founds another school, Pythagoras and the Italic school, in which it seems reasonable to place the fountain-head of the majestic river of geometric research. In fact, in it, with the stable systematisation attained by the theory of ratio and of proportion, the study of the problems of the "applications of areas" and the introduction of irrational quantities, were prepared the instruments which came afterwards to be continually used by the most eminent of the ancient geometers, and to which must always recur whoever intends to follow their luminous footsteps.

The destruction of the Pythagorean community did not extinguish the enthusiasm for mathematics among the admirers and followers of the philosopher of Samos; certainly we find in Hippocrates of Chios and Archytas of Tarentum, late disciples of Pythagoras, two strenuous workers in the research of geometric truth. Nor did the other philosophical sects which were afterwards so numerous in Greece remain indifferent to the progress of exact science: this is shown by what we know respecting Zeno and Democritus, Anaxagoras and Hippias, Plato and Aristotle, and the phalanx of thinkers who were instructed or directed by these.

Thanks to the co-operation of so many great geniuses, there was laid so solid a foundation for the geometric edifice, that more than one judged that the time had come to organise into a body of doctrine the results of the investigations accomplished. Besides were studied thoroughly the three famous problems, the quadrature of the circle, the duplication of the cube, and the trisection of the angle, which gave opportunity (to Archytas, Eudoxus of Cnidos, and Deinostratus), for inventing important new lines plane and of double curvature. Moreover there were discovered and coordinated by Aristæus the Elder, the properties and various applications made of those curves which Menæchmus discovered and Kepler was later to recognise as the trajectories of the planets; finally the conception of infinity makes timidly its ingress into mathematics, where it was destined to occupy later a position of exceptional importance. At the same time also the methods of research and exposition of geometric truth were made objects of accurate study;—we arrive thus at the *method of reduction* due to Hippocrates, the *analytical method* formulated by Plato, the *method of exhaustions* so brilliantly applied by Eudoxus. On the other hand by introducing the diorism Leon points out an important complement required in the solution of any problem; while by determining the conditions of invertibility of a theorem, Menæchmus teaches a fruitful method for deducing from one proposition, others. Moreover, logic which is joined to mathematics by so many intimate bonds, received powerful impulses from dialectics, from the sophists, and from the teaching of Socrates; it was in consequence so conspicuously perfected, that Aristotle thought the moment had come to expound its fundamental canons in a work which afterwards remained a classic for long ages.

4. From this appears how everything seemed to conspire to the emergence of a period of singular bloom for geometry, and in fact this did not delay its coming; this falls within the Græco-Alexandrine epoch and, compared with the periods preceding and following, merits the name of *golden period of Greek geometry*, by which we choose to designate it. To make a completely faithful picture of it is to-day an impossible undertaking,[1] since only a few persons of that epoch can be portrayed with exactness; of others we perceive only a few outlines, of still others we know the existence, but are not able to descry the lineaments and contour. However, what we know enables us to assert that, just as Greek philosophy in the period of its most dazzling splendor found in Socrates, Plato and Aristotle its most conspicuous representatives, so in the golden period of Greek geometry tower gigantic Euclid, Archimedes, and Apollonius. Through the merit of the first of these geometers the civilised world came into possession of a wisely arranged collection of the most essential properties of figurate space, a collection which throughout centuries and centuries was consid-

---

[1] See G. Loria, *Le scienze esatte nell' antica Grecia*, Libro II, *Il periodo aureo della geometria greca*. (*Mem. della R. Accad. di Modena*, II, 11, 1895).

ered a code of insuperable value and which always inspires admiration and respect even in those who do not blindly accept its arrangement and precepts. The second—Archimedes—founder of the family of Italian geometers, organiser of the higher metrical geometry, precursor of Leibnitz and Newton—reveals himself to us as of such marvellous fecundity in devising expedients to solve, entirely without the aid of the concept of infinity, a number of questions which to-day are regarded as strictly pertaining to the infinitesimal calculus, that the study of these still fills us to-day with astonishment and makes us sadly ask ourselves whether the invention of general methods which has so much exercised modern scientists has not perhaps dried up the source of ingenious expedients. Less vivid and spontaneous does the admiration arise in one who contemplates the works of Apollonius, because we are so identified with the present methods of research, quick and general, that with difficulty can we appreciate how great a mass of work was required in reaching the truth without invoking their aid; thus we can hardly defend ourselves from a sense of surprise which retards the enthusiasm; but when we attain this we come to justify those who consider Apollonius the greatest geometrical genius that the world had seen before Steiner.

The united efforts of these three great mathematicians, of their contemporaries or immediate successors, (among whom we name Hypsicles of Alexandria, Diocles, Nicomedes, Perseus, and Zenodorus), established an impregnable foundation for the whole geometric edifice, prepared the ground for the infinitesimal calculus and increased immeasurably the sphere of influence of geometry by bringing within its confines new and most interesting geometric forms. These led to numerous solutions of the problems (already famous in the pre-Euclidean period) of the duplication of the cube and the trisection of the angle, and taught geometers to consider that of the quadrature of the circle from the only point of view which—with the algebraic knowledge of the epoch—then permitted a solution. Moreover these laid the foundation of the geometric study of maxima and minima, and particularly of the doctrine of isoperimetry. Lastly by furnishing the most important elements of that celebrated didactic aggregate which goes by the name of *locus resolutus*, they employed themselves in paving the way for future investigators, striving thus that the spirit of geometric research should not die with them.

5. The list of great names which Greek mathematics vaunts would present an unpardonable gap if it did not mention Heron of Alexandria and Claudius Ptolemy.[1] The scientific work of the former was specially geodetic and that of the second, astronomic; this makes Heron appear as the most ancient Greek writer on geodesy, and establishes his close connection with Ahmes (see § 1), but his work can be estimated with slight exactness and great difficulty on account of the present condition of the manuscripts, which are blemished by omissions and corrupted by additions of doubtful value made by anonymous commentators;[2] on the contrary the achievements of Ptolemy can be judged because his most important work—the *Mathematical Composition* (*Syntaxis Mathematica*) or *Almagest*—has reached us (by way of the Arabs) complete, and presents to us a treatise of spherical trigonometry accompanied by important chapters on plane trigonometry and founded in part on the theory of transversals for plane and spherical triangles, of which the geometer Menelaus of Alexandria had previously laid the foundations.

After the great thinkers mentioned in this and the preceding paragraph, for many intrinsic and extrinsic reasons, which this is not the place to discuss, the love for geometric research seems to grow cold among the Greeks, or at least the power to accomplish it disappears; then begins the epoch of the commentators, which we choose to call by the name of *silver period of Grecian geometry.*[3]

---

[1] Cf. G. Loria, *Le scienze esatte nell' antica Grecia*, Libro III., *Il substrato matematico della filosofia naturale dei Greci* (*Mem. dell' Accademia di Modena*, 1900).

[2] The recent discoveries of the Μεπρίνα made in a manuscript now at Constantinople and the complete edition of the works of Heron which is now being prepared lead us to hope that soon upon this scientist also we shall be able to pronounce a definite judgment.

[3] Cf. G. Loria, *Le scienze esatte nell' antica Grecia*, Libro IV., *Il periodo argenteo della geometria greca* (*Mem. dell' Accademia di Modena*, 1900).

To this appertain Eutocius and Proclus, whom we gratefully remember on account of the numerous and important points of information which they preserved for us concerning the ancient geometry and the ancient geometers, but who did not carry our science a single step forward; rather we might say that Proclus tried to make it retrograde, when in commenting on Euclid he mixed mathematics and philosophy, as if he wished to replace in honor a system which precisely Euclid, perhaps the first, had with full reason vigorously combated by his example. Neither does Serenus of Antissa or Antinoupolis[1] merit much greater honor, who dealing with the sections of the cylinder incidentally demonstrated that the harmonic relation is projective, nor Theon of Alexandria who edited Euclid and illustrated Ptolemy. But let us bow reverentially to Pappus of Alexandria who brought notable contributions to the works on which he commented in his celebrated *Mathematical Collections*; in it noteworthy additions to elementary geometry are made, new properties of the spiral of Archimedes are pointed out, two new constructions for the Quadratrix of Deinostratus which are extremely noteworthy as being founded on stereometric considerations are demonstrated, and the spherical curve analogous to the spiral of Archimedes is defined and studied, which gave Pappus the occasion for finding the area of a certain portion of the sphere, thus making the first known complanation of an area not plane.

We pass over the additions made to the theory of isoperimeters to record the theorem unjustly attributed to Guldinus (1577–1643) establishing a close relation between the position of the baricenter of a plane figure and the volume which this generates by rotation about an axis; furthermore the numerous relations of segments which are to-day considered as indispensable parts of the theory of harmonic relation and of involution. Does not this suffice to show that the Greek mathematical genius, like a failing lamp, emits, while dying, a beam of resplendent light?

6. On the mathematical stage as prime actor the Roman peo-

---

[1] See Heiberg, "Ueber den Geburtsort des Serenos" (*Bibl. math.*, 1894).

ple follows the Greek, but its appearance marks an indisputable retrogression.[1] For the Romans, conquerors and legislators of the whole world, seem to have been without any tendency toward research in pure science.[2] Their mathematics, even at the moment in which it most shone, was essentially practical; we could even say that it was only religious and legal.[3] If geometry did not during the Roman epoch fall into complete oblivion, the merit is due to the surveyors, who, however, had no other intention than to reach in their operations an accuracy sufficient for the every-day needs of civil life, and in particular to execute the order of Augustus, who wished to have carried out the ancient project of Julius Cæsar to measure the area of the empire.[4]

The Middle Ages do not afford material for long consideration. The utter darkness which during this period wrapped up all mankind is not pierced by any sprinkling of light derived from any scientist worthy of the name; to prove how low was then the level of mathematical knowledge, it is sufficient to remark that among all the students of that epoch, Gerbert (d. 1003)—later Pope Sylvester II.—appears a giant, and yet we try in vain to cite some discovery of his in the field of exact science. However, we may note that the numerous sacred monuments erected during the Middle Ages—which according to the genial conjecture of a great poet are so numerous and bold because they represent the only method of expression then conceded to the human intellect[1]—prove that such part of the theoretic science as every architect must have at his disposal was even in those times generally known. But a word of grateful praise must be spoken in favor of the Arabs, who performed the modest, but useful and certainly not ignoble, part of transmitters into Europe of the Hellenic science, nor did they fail to bring to our mathematical knowledge (especially in the arithmetical field) additions and modifications worthy of being honorably mentioned in the annals of our science.[2]

---

[1] To confirm this assertion, we cite the following eloquent words of the celebrated historian of Italian mathematics: "....mais bientôt le Romain arrive, il saisit la science personifiée dans Archimède, et l'étouffe. Partout où il domine, la science disparaît: l'Étrurie, l'Espagne, Carthage en font foi. Si plus tard Rome n'ayant plus d'ennemi à combattre, se laisse envahir par les sciences de la Grèce, se sont des livres seulement qu'elle recevra: elle les lira et les traduira sans y ajouter une seule découverte. Guerriers, poètes, historiens, elle les a eus, oui; mais quelle observation astronomique, quel théorème de géométrie, devons-nous aux Romains?" Libri, *op. cit.*, I., p. 186.

[2] Cicero himself recognised this, saying: "In summo honore apud Græco-geometria fuit; itaque nihil mathematicis illustrius; at nos ratiocinandi metiens dique utilitate huius artis terminavimus modum." *Tusc.* 1, I., c. II., 5.

[3] To show how the Romans misunderstood the true character of mathematics, suffice it to say that they confounded it with astrology and the sister arts; no wonder then that in the code of Justinian certain laws are collected under the title: *De maleficis et mathematicis et ceteris similibus*, and among them the following: "ars autem mathematica damnabilis interdicta est omnino." See M. Cantor, *Mathematische Beiträge zum Kulturleben der Völker* (Halle, 1863), p. 397; Hankel, *Zur Geschichte der Mathematik im Alterthum und Mittelalter* (Leipzig, 1874), p. 301. It is well to observe also that if during the imperial period we meet some mathematical investigation, its author is some Greek.

[4] It is expressly to the practical part of geometry that the Roman legislator alludes when in the above-mentioned Code he wrote: "Artem geometriae discere atque exercere publice interest." (See Hankel, *l. c.*)

7. This period, so sad for the mathematic discipline, can be said to end about 1200 with the appearance of Leonardo Fibonacci of Pisa (1180–1250 *circa*), able calculator, fine geometer, and genial algebraist; for only when algebra, transported into Europe by this eminent scientist, and his meritorious works had exercised their influence—and to perceive it manifest we must go down to the appearance in 1494 of the *Summa* of brother Luca Paciuolo (about 1445–1514)—commenced a time of extraordinary activity, which Italians mention with legitimate pride, because then that country swayed worthily the mathematical sceptre. But we observe that this period, if it had great importance for analytical research, did not assist in any sensible progress of our geometrical thought. In truth Tartaglia (1500–1557) and Cardan (1501–1576), Scipio Ferro (?–1526) and Lodovico Ferrari (1522–1565), and the other minor geometers of that time have the glory to have contributed effec-

---

[1] See the admirable chapter entitled "Ceci tuera cela" of *Notre Dame de Paris*, where Victor Hugo writes among other things: "Il existe à cette epoque, pour la pensée écrite en pierre, un privilège tout à fait comparable à notre liberté actuelle de la presse. C'est la liberté de l'architecture."

[2] See Suter, "Die Araber als Vermittler der Wissenschaften in deren Uebergang vom Orient in den Occident." (*XXV. Jahresheft des Vereins schweiz. Gymnasiallehrer*, 1895.)

tively to the foundation and development of the theory of equations, not disdaining to propose and accept the public challenges, which were a noteworthy characteristic of this epoch; but, on the other hand, they transmitted to posterity geometry very nearly as they received it from the Greeks by way of the Arabs. Only we observe that, among the questions dealt with in that epoch by the Italian scientists, some refer to the geometrical constructions to be executed with a single opening of the compasses. This is a conception which is unjustly considered by Montucla as *une bagatelle géométrique*,[1] which is found in Number V. of the challenge problems sent in the celebrated contest between Tartaglia and Ferrari in October, 1547,[2] and which G. B. Benedetti (d. 1590) has unfolded in his work *De resolutione omnium Euclidis problematum aliorumque una tantummodo circuli data apertura* (Venice, 1553) and which gave rise later to *La geometria del compasso* (Pavia, 1797) by Lorenzo Mascheroni (1750–1800)—in which is demonstrated the possibility of solving without a straight edge all the problems of elementary geometry—and, after having been modified, to *Die geometrischen Constructionen ausgeführt mittelst der geraden Linie una eines festen Kreises* (Berlin, 1833) by Jacob Steiner (1797–1863)—a work in which is indisputably demonstrated a fact (that indicated by the title) which, as this great geometer recognises and indicates, had already been pointed out.[3]

8. After the time of the able scientists to whose labors we owe the solution of the equations of the third and fourth degree, the primacy in mathematics was attained by France, especially through the merits of Viète (1540–1603), whose labors for algebra it is not necessary to record here, and whose interest for geometry is shown

---

[1] *Histoire des mathématiques.* T. I. (2d ed. Paris An. VII.), p. 570.

[2] For greater particulars see Kutta, "Zur Geschichte der Geometrie mit constanter Zirkelöffnung" (*Nov. Acta Abh. der k. Leop. Carol. Akad.*, 71, 1897). It is sufficient to note that Ferrari himself makes the consideration of such problems go back to a period much anterior; in fact, he speaks of "that fine invention, to operate without changing the opening of the compasses, of which I do not know the origin, but which I know well that many fine geniuses have in the last fifty years worked hard to add to."

[3] Poncelet, *Traité des propriétés projectives des figures* (Paris, 1822) §§ 351–357.

by the construction which he gave for a circle tangent to three others, expounded by him in the form of a restoration, in the work entitled *Apollonius Gallus seu exsuscitata Apollonii Pergaei* περὶ ἐπαφῶν *geometria* (Paris, 1600). Not long afterwards Mydorge (1585–1647), Pascal (1623–1662),[1] and Desargues (1596-1662)[2] strengthened the supremacy of France by adding to the patrimony of geometry original views, new methods, and new propositions; in fact in *Claudii Mydorgi Conicorum operis* etc. (Paris, 1631–1639) is found, not to mention other things, for the first time the homothetic transformation of one conic into another: from the *Traité général de la Roulette* of Pascal we learn a number of admirable properties of the cycloid, while the applicability of the method of perspective to the study of the conic sections and the celebrated theorem which bears the name of Pascal (although the author had preferred to designate it by that of the "mystic hexagram") are found in the celebrated *Essai pour les coniques;* finally in the *Œuvres de Desargues* (ed. Poudra, Paris, 1864) are found for the first time treated together the three conic sections, the concept of the involution of points is introduced, and its presence demonstrated in the section of a plane quadrilateral with a right line ("transversal"), and not only is perspective applied, but also with great breadth and ability the idea of infinity[3] as a potent auxiliary for the research of the plastic properties of space.[4]

That also in England and Germany geometry had not fallen during this epoch into a complete oblivion is shown by the lectures given at Oxford by Henry Savile (1549–1622), who founded there two chairs of mathematics still in existence, and by the geometric researches of John Kepler (1571–1630) to whom we can attribute[1] the concept of the point at infinity of the straight.

---

[1] See J. Bertrand, *Blaise Pascal*, (Paris, 1891).

[2] S. Chrzaszewski, "Desargues's Verdienste um die Begründung der projectivischen Geometrie" (*Archiv*, II., 16, 1898).

[3] Newton inspired by Desargues wrote: "Lineae parallelae sunt quae ad punctum infinite distans tendunt." *Principia*, Book I., Lemma 22.

[4] To Desargues Poncelet attributes the geometrical method which consists in substituting for a curve a system of straights (*Propr. proj.*, 2nd ed., Vol. II., Paris, 1866, p. 128); but that this attribution, although accepted by many, lacks a solid foundation, I have proved in the note entitled "Desargues e la geometria numerativa" inserted in the *Bibl. math.*, 1895.

9. The greater part of the geometric ideas given and applied by the geometers just cited remained for long years fruitless. They were as if suffocated in the heavy analytical atmosphere which we have already indicated. Yet in the seventeenth century the supremacy of analysis is not yet so manifest as to make geometers forget the problems whose solution they had so long and so earnestly desired; whence it is that then among the tendencies of the epoch and the aspirations of the scientists there arose a contest *sui generis*, by the clash of contradictory opinions was generated a spark, mother of a conflagration destined to illuminate the entire universe[2] and

> "Di cui la fama ancor nel mondo dura
> E durerà quanto il moto lontana:"

—analytical geometry is born.

Although already in some practical proceedings employed by artists and astronomers of Egypt and later by the Roman surveyors, as well as in some methods which were applied by Greek geometers of the first order, such as Heron and especially Apollonius,[3] it is easy to see some trace of what to-day we call the system of orthogonal Cartesian co-ordinates ; although already the Arabs and the Italian algebraists of the Renaissance had adopted geometric considerations to solve graphically certain equations; although Vieta had already used abscissas for particularising with the help of numbers the points of a straight, and Nicole Oresme (about 1320–1382) had made use more or less explicitly of co-ordinates it seems now settled beyond dispute that Descartes (1596–1650)[1] and Fermat (1608–1655)[2] were the first to see in its whole extent the possibility of representing by means of the symbols of algebra the forms of space constructed according to any law, and to perceive all the advantage which analysis and geometry could derive from their unexpected marriage.[3] Fermat perhaps came before Descartes to the new branch of mathematics, but what he published is of later date than the celebrated *Géométrie* (1637). In this work is given explicit indication of the equation of the curve, and a solution of the problem *ad tres aut quatuor lineas*, so famous among the ancients; moreover the distinction between algebraic and transcendental curves is suggested, and a classification, now abandoned, of the algebraic curves is proposed, by adopting which the curves of the degrees $2n-l$ and $2n$ would be classed in the same *genus*; a method is taught of drawing tangents to plane curves, which later is applied to the conchoid and to the ovals of Descartes; nor is the possibility overlooked of extending to space the method of co-ordinates. On the other hand, in the memoir by

---

[1] See C. Taylor, *Ancient and Modern Geometry of Conics* (Cambridge, 1881), pp. lvii–lix.

[2] See E. du Bois-Reymond, "Culturgeschichte und Naturwissenschaft" in the collection *Rede*, I., 1886, pp. 207–208.

Concerning the prehistory of analytical geometry see Günther, "Die Anfänge und die Entwickelungsstudien des Coordinatenprincips" (*Abhandl. der Nat. Ges. zu Nürnberg*, 6; compare *Bull. Bonc.*, 10); the conclusions of this work were confuted by Zeuthen in the "Note sur l'usage des coordonnées dans l'antiquité et sur l'invention de cet instrument (*Bull. de l'Acad. danoise des Sciences*, 1888), to which Günther replied in the *Neue philosophische Rundschau*, 1888.

[3] In truth, whoever studies thoroughly the treatise of Apollonius on *Conics* must confess the profound analogy it bears to an exposition of the properties of the curves of the second degree by means of Cartesian co-ordinates; not only do the fundamental properties employed by the Greek geometer to distinguish the three curves one from the other translate into the canonical equations of the same in Descartes's method, but many of the reasonings given, when translated into the ordinary language of algebra, answer to elimination, solution of equations, transformation of co-ordinates, and the like. What we would however seek in vain in the Greek geometer is the concept of a system of axes, given *a priori*, independent of the figure to be studied.

---

[1] See Jacobi, "Ueber Descartes's Leben und seine Methode, die Vernunft richtig zu leiten und die Wahrheit in den Wissenschaften zu suchen" (*Ges. Werke*, 7, 1891; see also *Journ. de Math.*, 12, 1847), Arago, *Œuvres Complètes*, 3, Paris, 1855.

[2] Arago, *loc. cit.* See *Œuvres de Fermat* (ed. P. Tannery and C. Henry), 1, (Paris, 1891), 2 (1894) and 3 (1896).

[3] That the *Analysis geometrica, sive nova et vera methodus resolvendi tam problemata geometrica quam arithmetica quaestiones*, by Hugo de Omerique (1634–?), the first part of which appeared at Cadiz in 1698, suffices to make its author seem the precursor of modern analytical geometry (as is affirmed by P. A. Berenguer in the article entitled "Un geómetra español del siglo XVII.," inserted in *Progréso*, 5, 1895), is a yet unsolved question, because for this purpose it would be necessary to know the second part of that work, which has not yet come to light.

Fermat *Ad locos planos et solidos isagoge* (in *Varia Opera*, 1679) the conception of the equation of a curve is more clearly set forth than in the *Géométrie*; besides, the equation of the straight is employed, the general equation of the second degree with two indeterminates is discussed, and the method of co-ordinates is applied to the solution of equations. If we add that Fermat discovered and corrected some errors of Descartes, it will be seen that it is rather by a lucky chance than on account of true merit that the discovery of analytical geometry is regarded as inseparably connected with the name of the author of the *Discours de la méthode*, instead of with that of Fermat.

The facility with which the new mathematical instrument solved problems which had surpassed the powers of the ancients, caused the majority of Descartes's and Fermat's contemporaries and immediate successors to abandon the path opened up by Euclid and by Apollonius; they preferred to enter upon the new road which, on the one hand, appeared smooth and easy to travel,[1] and, on the other hand, attracted them on account of the obstacles with which it was always bestrewn, and which seemed very important to remove. Among these early cultivators of analytical geometry are conspicuous J. Wallis (1616–1703) for his *Tractatus de sectionibus conicis nova methodo exposita* (1655), Florimond de Beaune (1601–1652), and Franz van Schooten (?–1661) thanks to their commentaries on Descartes, and still more Johann De Witt (1625–1672) for his *Elementa curvarum linearum* (1658).

10. The modes of procedure of mathematical investigation founded on the methodical use of the infinite and the infinitesimal, applied by Archimedes in antiquity in a peculiar manner and prepared many centuries later by Kepler's *Stereometria doliorum* (1615) and the *Geometria indivisibilibus, etc.* (1635) of Bonaventura Cavali-

---

[1] "Ainsi, à certaines époques, où, après de grands efforts, les sciences mathématiques semblent avoir épuisé toutes les ressources de l'esprit humain et atteindre le terme marqué à leurs progrès, tout à coup une nouvelle méthode de calcul vient s'introduire dans ces sciences et leur donner une face nouvelle. Bientôt on les voit s'enrichir rapidement par la solution d'un grand nombre de problèmes importants dont les géomètres n'avaient osé s'occuper, rebutés par la difficulté." Condorcet in his *Eloge de M. Euler*.

eri (1598–1647), by the methods of constructing tangents to curves imagined by Descartes and by those which Fermat suggested for the research of maxima and minima,[1] finally by the investigations of Roberval (1602–1675)[2], of Torricelli (1608–1647, *Opera geometrica,* 1664)[3] of Gregory St. Vincent (1584–1667) (*Opus geometricum*, 1647), of Wallis (*Arithmetica infinitorum*, 1655), of Pascal, etc., and invented, soon after the appearance of the *Géométrie* of Descartes, at the same time by Leibniz (1646–1716)[4] and Newton (1642–2727),[5] emphasised the tendency characterised in the preceding section, causing all those problems to be neglected whose solution was not adapted to exhibit the omnipotence of the methods which the world owes those immortal minds. So much so that we may say that, excepting Newton's *Philosophiae naturalis principia mathematica* (1686) and some hints of Leibniz[6] of a "*Characteristica geometrica,*" a few pages of Huygens (1629–1695)[7] and of Maclaurin (1698–1746)[8] as also, finally, some memoirs by Stewart (1717–1785)[9] and by de la Hire (1640–1718)[1] and a few efforts to revive the ancient geometry,—efforts to which we shall immediately devote a few lines,—no eminent mathematical production of the epoch of which we speak belongs to pure synthetic geometry.

---

[1] See Zeuthen's interesting note "Sur les quadratures avant le calcul intégral et en particulier sur celles de Fermat" (*Bulletin de l'Acad. danoise des Sciences*, 1895).

[2] *Mem. de l'Acad. des Sciences*, 6, Paris, 1630.

It is important to bear in mind that some inaccuracies in the *Observations sur la composition des mouvements, et sur les moyens de trouver les touchantes des lignes courbes* by Roberval were rectified by Duhamel (1797–1872) in the "Note sur la méthode des tangentes de Roberval" (*Mem. des Sav. Étr.*, 5, Paris, 1838).

[3] See also: G. Loria, "Evangelista Torricelli e la prima rettificazione di una curva" (*Lincei Rend.*, V., 6, 1897, ii.), and "Le ricerche inedite di Evangelista Torricelli sopra la curva logaritmica" (*Bibliotheca mathematica*, III., 1, 1900).

[4] See Fontenelle, *Éloges des académiciens* (La Haye, 1740), 1; Harnack, *Leibniz' Bedeutung in der Mathematik* (Dresden, 1887).

[5] See Fontenelle, *op. cit.*, 2; Arago, *Œuvres complètes*, 3 (Paris, 1855); Brewster, *The Life of Sir I. Newton* (London, 1831), and *Memoirs of the Life, Writings and Discoveries of Sir I. Newton* (Edinburgh, 1855).

[6] *Mathematische Schriften* (ed. Gerhardt), 5, pp. 141–171.

[7] See in particular *Horologium oscillatorium* (Paris, 1673), and *Traité de la lumière* (Leyden, 1691); moreover his scientific correspondence published in *Œuvres complètes de Chr. Huygens* (La Haye, 1888 and follow.).

[8] *A Complete System of Fluxions* (Edinburgh, 1742).

[9] *General Theorems of Considerable Use in the Higher Parts of Mathematics* (Edinburgh, 1746); see its analysis by Breton (de Champ) in the *Journal de*

The attempts to which we have just now alluded are found, we may say, in every century, beginning from the Renaissance, and appear either in the form of editions of the ancient geometers and commentaries upon them, or in that of restorations of works which were not saved from the ravages of time. It is not here in place to speak of the editions and commentaries, but we must devote some lines[2] to the reconstructions, in the first place because their significance is indisputable when they bear the names of Maurolycus (1494–1575), Vieta, Fermat, Schooten, Viviani (1622–1703), Halley (1656–1742), Simson (1687–1768),[3] Horsely (1733–1807), and in the second place because their appearance is the first indication of an awakening of the spirit of geometrical research. And in truth this is a phenomenon which seems to us worthy of the most serious consideration that in such a state of somnolence as preceded the awakening, scientists did not succeed in resisting the fascination exercised by the most ancient investigators, and, in consequence, attempted to tread in their foot-steps and even to imitate their movements and their gestures. We should therefore be led to conjecture

---

Math., 13, 1848, as well as a valuable article on the latter by T. S. Davies, in the *Trans. of Edinburgh*, 15, 1844); *Propositiones geometricae more veterum demonstratae* (Edinburgh, 1763).

[1] See—besides Fontenelle, *op. cit.*, 2—E. Lehmann, *De la Hire und seine sectiones conicae*, I. Th., Leipzig, 1888; II. Th., Leipzig, 1890. Of De la Hire we mention here the following writings: *Sectiones conicae in novem libros distributae*, 1685; *Mémoirs sur les épicycloïdes*, in *Anc. Mém. de Paris*, 9; "Traité des roulettes" in *Nouv. Mémoirs de Paris*, 1706; *Des conchoïdes générales*, Id., 1709.

[2] More particulars will be found in the appendices to the first two books of my work, cited above, on *Le scienze esatte nell' antica Grecia*.

[3] Concerning the efforts made by Simson and Stewart to give new life to ancient geometry, see Buckle's remarks in *History of Civilisation in England*, 4. Let us add that the fight in England between analysts and synthetists prolonged itself even into our century; this is proved by the two rival works: *Geometrical Analysis and Geometry of Curve Lines*, by J. Leslie (Edinburgh, 1821), and *A Treatise on Algebraic Geometry*, by D. Lardner (London, 1831).

that in the intellectual life of nations there happens something analogous to what embryology shows to take place in the physical life of every living being; in the same way that every organised being, before acquiring an autonomous and independent life, passes through all the phases of development that the species to which it belongs traversed before reaching the physical state of the generation to which it shall belong, so it would appear that every people before acquiring the capacity of increasing our knowledge concerning the phenomena which are presented by figured extention, must for some time assume the appearance and the ways of acting of those who before them went over the same road. This law of development being accepted as true, we avoid including among the rubbish that which instead is a germ, and we can render an account of certain facts which otherwise would be considered surprising anachronisms. And among these none seems more worthy of mention than that presented by the "Neapolitan School" so much extolled by Chasles,[1] which had as supreme leader Nicola Fergola (1753–1822) and as secondary heads or privates Annibale Giordano (1771 *circa* 1835), Vincenzo Flauti (1782–1863), Felice Giannattasio (1759–1849), Guiseppe Scorza (1781–1843), and others whose names are omitted for brevity;[2] a school which would be placed among those which had very little influence on geometry if it should not instead, according to our view, be retained to represent a stage which the mathematics of the noon-tide of Italy had necessarily to go through before being ready to combat for the conquest of new truths.

11. But it is necessary that we turn back to determine what may have been the influence on the development of geometry of the fundamental concept of infinitesimal analysis, which is considered by a great philosopher and mathematician as the most sublime thought to which the human mind has attained up to the present time.[3] We cannot deny that its appearance had drawn away the

---

[1] *Aperçu historique*, 2nd ed. (Paris, 1875), p. 46

[2] For further notice refer to my work entitled *Nicola Fergola e la scuola dei matematici che lo ebbe a duce* (Genoa, 1892).

[3] Comte, *Cours de philosophie positive*, 1 (Paris, 1864), p. 175.

majority of mathematicians from pure geometry;[1] nevertheless the period which comes after this must without hesitation be counted among the happiest for geometry, since the greater part of the problems proposed or solved by Newton and Leibniz and by their immediate disciples must be placed among the most important embraced by geometry, relating to the most recondite and interesting geometrical or mechanical properties of curves and surfaces. As a consequence we see extraordinarily augmented not only the number of curves made the objects of study,[2] but (what is more important) we see introduced the consideration of the singularities of a curve and of new elements connected with this, and we see revealed in consequence fields of research of which no one before had imagined the existence.

The facilities afforded by the Cartesian method for investigations concerning the geometry of the plane, which were at their height when the infinitesimal method came into the possession of the public, naturally compelled scientists to seek an analogue to it for the study of tortuous curves and of curved surfaces. Hence the stimulus to the generalisation for space of the method of co-ordinates which, as we said, did not escape Descartes, and which Schooten noted explicitly in the last book of his celebrated *Exercitationum mathematicarum* (1657) ; hence the origin of the idea of Parent (1666–1716)[1] to represent any surface by means of an equation between the three co-ordinates to any of its points.[2]

---

[1] This brings a new confirmation of the remark of Condorcet which we referred to in the note at the conclusion of the preceding paragraph.

[2] The lines which the Greeks knew were—omitting the straight and the circle—the three conic sections, the spiral of Archimedes and the analogous spherical curve, the conchoid, the cissoid, and a quadratrix, besides a tortuous curve of Archytas, the hippopede of Eudoxus, and the cylindric helix; to which others may be added when certain historical questions still pending have been solved in a sense favorable to the ancients. To these lines the moderns have added various species of hyperbolas, of parabolas, of spirals (for instance the logarithmic) and of quadratrices (among which is conspicuous that of Tschirnhausen), then the folium and the ovals of Descartes, the cycloid, (see Günther in *Bibl. math.*, 1887), the loxodrome, the elastic curve, the catenary, the cardioid, the epicycloid and hypocycloid, the ovals of Cassini (in particular the lemniscata), the sinusoid, the logarithmic curve, the *versiera* of Maria Gaetana Agnesi (1718–1799) (*Intituzione analitiche*, Milano, 1748, I, p. 380), certain plane and spherical curves at which Guido Grandi (1671–1742) arrived while trying to solve the famous Florentine enigma (see the work: *Flores, geometrici*, etc., Florence, 1728), then others which Vincenzo Viviani and a friend of his proposed as auxiliary in the solution of the problem of Delos (see *Quinto libro di Euclide, ovvero scienza universale delle proporzioni spiegata colla dottrina del Galileo da V. V.*, Florence, 1647), the parabolic spiral, (see Weyer, *Ueber die parabolische Spirale*, Kiel und Leipzig, 1894), etc., etc. For greater details see Brocard, *Notes de bibliographie des courbes geometriques* (Bar-le-Duc, 1897 and 1899).

This period of preparation of analytical geometry of space is followed by the first period of development, in which figured as leaders Clairaut (1715–1765) and Euler (1707–1783).[3] Clairaut in 1731, when only sixteen years old, published a *Traité des courbes à double courbure*, that was considered worthy of being admired as a marvel of imagination and of ability, in which are solved with rare elegance many of those problems relative to the curves of double curvature which have their analogues in the plane, and others then entirely new. Euler supplied an *Appendix de superficiebus* to the second volume of his celebrated *Introductio in analysin infinitorum* (Lausannae, 1748), in which appendix he not only gave some general propositions concerning the analytical treatment of the theory of surfaces, but made an important application of it to the study and classification of the quadrics. This same famous scientist by the memorable "Recherches sur la courbure des surfaces" (inserted in the *Mémoires de l'Académie des Sciences de Berlin*, 16, 1760) laid the foundations for the study of the curvature of a surface in one of its points, giving his own name to a relation which, by the continual use made of it, is very familiar to all those who are even slightly occupied with mathematics.

To the second half of the eighteenth century belongs also in a large measure the gigantic scientific work of Monge (1746–1818),[4] who, after having given to the analytical geometry of the plane the aspect which it retains in great measure to-day, using systematically the equation of the straight line, introduces the idea so important of *family of surfaces*, and, studying some special families ("Surfaces moulures," ruled, developable, tubular, etc.), disclosed to view a connection intimate and unexpected, because remote, between the theory of surfaces and the integration of partial differential equations, which, illumining both doctrines, revealed to geometers new refulgent horizons.

---

[1] Fontenelle, *op. cit.*, 1.

[2] *Essais et recherches de mathématique et physique*, 2 (Paris, 1713).

[3] N. Fuss., *Éloge de M. Euler* (St. Petersburg, 1783).

[4] I cite here the works inserted in the *Mém. des Sav. étrangers*, 7, 1776 and 9, 1780, and in the *Mém. de Paris*, 1784, and also those which are found in the *Miscellanea Taurinensia*, 5, and in *Mém. de l'Acad. de Turin*, 1. Of the *opus magnum* summarising and completing such works, we shall not speak here.

G. LORIA.

UNIVERSITY OF GENOA.

(TO BE CONTINUED.)